QUADRATIC RELATIONS FOR A q-ANALOGUE OF MULTIPLE ZETA VALUES

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Abstract. We obtain a class of quadratic relations for a q-analogue of multiple zeta values (qMZV's). In the limit $q \to 1$, it turns into Kawashima's relation for multiple zeta values. As a corollary we find that qMZV's satisfy the linear relation contained in Kawashima's relation. In the proof we make use of a q-analogue of Newton series and Bradley's duality formula for finite multiple harmonic q-series.

1. Introduction

In this paper we prove quadratic relations for a q-analogue of multiple zeta values (qMZV's, for short). The relations are of a similar form to Kawashima's relation for multiple zeta values (MZV's).

First let us recall the definition of qMZV [1, 11]. Let $\mathbf{k} = (k_1, \ldots, k_r)$ be an r-tuple of positive integers such that $k_1 \geq 2$. Then qMZV $\zeta_q(\mathbf{k})$ is a q-series defined by

$$\zeta_q(\mathbf{k}) := \sum_{m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}, \tag{1.1}$$

where $[n]$ is the q-integer

$$[n] := \frac{1-q^n}{1-q}. \tag{1.2}$$

Since $k_1 \geq 2$, the right hand side of (1.1) is well-defined as a formal power series of q. If we regard q as a complex variable, it is absolutely convergent in $|q| < 1$. In the limit as $q \to 1$, qMZV turns into MZV defined by

$$\zeta(\mathbf{k}) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

An interesting point is that qMZV's satisfy many relations in the same form as those for MZV's. For example, qMZV's satisfy Ohno's relation [6], the cyclic sum formula [3, 7] and Ohno-Zagier's relation [8]. See [1] for the proof of Ohno's relation and the cyclic sum formula for qMZV's, and see [9] for that of Ohno-Zagier's relation.

In this paper we prove a class of quadratic relations for qMZV's (see Theorem 4.6 below). In the limit as $q \to 1$, it turns into Kawashima's relation [4] for MZV's. Some of our relations are linear, and they are completely the same as the linear part of Kawashima's relation (Corollary 4.7). It is known that the linear part of

Kawashima's relation contains Ohno's relation [4] and the cyclic sum formula [10], and hence we find again that qMZV's also satisfy them.

The proof of our quadratic relations proceeds in a similar manner to that of Kawashima's relation. The ingredients are a q-analogue of Newton series and finite multiple harmonic q-series. Let $b=\{b(n)\}_{n=0}^{\infty}$ be a sequence of formal power series in q. Then we define a sequence $\nabla_q(b)$ by

$$\nabla_q(b)(n):=\sum_{i=0}^{n} q^i \frac{(q^{-n})_i}{(q)_i}\, b(i)$$

and consider the series

$$f_{\nabla_q(b)}(z):=\sum_{n=0}^{\infty} \nabla_q(b)(n)\, z^n \frac{(z^{-1})_n}{(q)_n},$$

where $(x)_n$ is the q-shifted factorial

$$(x)_n:=\prod_{j=0}^{n-1}(1-xq^j). \tag{1.3}$$

Under some condition for $\nabla_q(b)$, the series $f_{\nabla_q(b)}(z)$ is well-defined as an element of $\mathbb{Q}[[q,z]]$ and it satisfies $f_{\nabla_q(b)}(q^m)=b(m)$ for $m\ge 0$ (see Proposition 3.1). Thus the series $f_{\nabla_q(b)}(z)$ interpolates the sequence b, and can be regarded as a q-analogue of Newton series. It has a nice property:

$$f_{\nabla_q(b_1)} f_{\nabla_q(b_2)} = f_{\nabla_q(b_1 b_2)}. \tag{1.4}$$

Now consider the finite multiple harmonic q-series $S_{\mathbf{k}}(n)$ defined by

$$S_{\mathbf{k}}(n):=\sum_{n\ge m_1\ge\cdots\ge m_r\ge 1} \frac{q^{m_1+m_2+\cdots+m_r}}{[m_1]^{k_1}\cdots[m_r]^{k_r}}.$$

Any product of $S_{\mathbf{k}}$'s can be written as a linear combination of them with coefficients in $\mathbb{Q}[(1-q)]$. The duality formula due to Bradley [2] (see Proposition 2.1 below) implies that the coefficients are qMZV's in the expansion of $f_{\nabla_q(S_{\mathbf{k}})}$ at $z=1$. Therefore, by expanding $f_{\nabla_q(S_{\mathbf{k}})} f_{\nabla_q(S_{\mathbf{k}'})} = f_{\nabla_q(S_{\mathbf{k}} S_{\mathbf{k}'})}$ at $z=1$, we obtain quadratic relations for qMZV's.

The paper is organized as follows. In Section 2 we define finite multiple harmonic q-series and describe their algebraic structure by making use of a non-commutative polynomial ring. In Section 3 we define a q-analogue of Newton series and prove the key relation (1.4). In Section 4 we prove the quadratic relations and see that qMZV's satisfy the linear part of Kawashima's relation for MZV's.

In this paper we denote by $\mathbb{N}$ the set of non-negative integers.

2. Algebraic structure of finite multiple harmonic q-series

2.1. **Finite multiple harmonic q-series.** Let $\hbar$ be a formal variable and $\mathcal{C} := \mathbb{Q}[\hbar]$ the coefficient ring. Denote by $\mathfrak{h}^1$ the non-commutative polynomial algebra over $\mathcal{C}$ freely generated by the set of alphabets $\{z_n\}_{n=1}^{\infty}$. We define the *depth* of a word $u = z_{i_1} \cdots z_{i_r}$ by $\mathrm{dep}(u) := r$.

Let $\mathcal{R} := \mathbb{Q}[[q]]$ be the ring of formal power series in q. We endow $\mathcal{R}$ with $\mathcal{C}$-module structure such that $\hbar$ acts as multiplication by $1-q$. Denote by $\mathcal{R}^{\mathbb{N}}$ the set of sequences $b = \{b(n)\}_{n=0}^{\infty}$ of formal power series $b(n) \in \mathcal{R}$. Then $\mathcal{R}^{\mathbb{N}}$ is a $\mathcal{C}$-algebra with the product defined by $(bc)(n) := b(n)\, c(n)$ for $b, c \in \mathcal{R}^{\mathbb{N}}$.

For a word $u = z_{k_1} \cdots z_{k_r}$, we define $S_u, A_u, A_u^{\star} \in \mathcal{R}^{\mathbb{N}}$ by

$$S_u(n) := \sum_{n \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{q^{m_1+m_2+\cdots+m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}, \tag{2.1}$$

$$A_u(n) := \sum_{n \geq m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1+(k_2-1)m_2+\cdots+(k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}, \tag{2.2}$$

$$A_u^{\star}(n) := \sum_{n \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{q^{(k_1-1)m_1+(k_2-1)m_2+\cdots+(k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}, \tag{2.3}$$

where $[n]$ is the q-integer defined by (1.2). Setting $S_1(n), A_1(n), A_1^{\star}(n) \equiv 1$, we extend the correspondence $u \mapsto S_u, A_u, A_u^{\star}$ to the $\mathcal{C}$-linear map $S, A, A^{\star} : \mathfrak{h}^1 \to \mathcal{R}^{\mathbb{N}}$.

Let $\mathfrak{h}^1_{>0}$ be the $\mathcal{C}$-submodule consisting of non-constant elements, that is, $\mathfrak{h}^1_{>0} = \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k \geq 1} \mathcal{C} z_{i_1} \cdots z_{i_k}$. For a word $u = z_{k_1} \cdots z_{k_r} \in \mathfrak{h}^1_{>0}$, set $s_u, a_u \in \mathcal{R}^{\mathbb{N}}$ by

$$s_u(n) := \sum_{n+1 = m_1 \geq m_2 \geq \cdots \geq m_r \geq 1} \frac{q^{k_1 m_1+(k_2-1)m_2+\cdots+(k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}, \tag{2.4}$$

$$a_u(n) := \sum_{n+1 = m_1 > m_2 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1+(k_2-1)m_2+\cdots+(k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}. \tag{2.5}$$

Extending by $\mathcal{C}$-linearity we define two maps $s, a : \mathfrak{h}^1_{>0} \to \mathcal{R}^{\mathbb{N}}$. Note that if $w = z_i w' \, (w, w' \in \mathfrak{h}^1, i \geq 1)$ we have

$$s_w(n) = \frac{q^{i(n+1)}}{[n+1]^i} A^{\star}_{w'}(n+1), \qquad a_w(n) = \frac{q^{(i-1)(n+1)}}{[n+1]^i} A_{w'}(n). \tag{2.6}$$

We define a map $\phi : \mathfrak{h}^1 \to \mathfrak{h}^1$ as follows. For a word $u = z_{k_1} \cdots z_{k_r}$, consider the set $I_u = \{\sum_{i=1}^{j} k_i \,|\, 1 \leq j < r\}$. Denote by I_u^c the set consisting of positive integers which are less than or equal to $\sum_{i=1}^{r} k_i$ not belonging to I_u. Set $I_u^c = \{p_1, \ldots, p_l\} \, (p_1 < \cdots < p_l)$ and define $\phi(u) := z_{k'_1} \cdots z_{k'_l}$, where $k'_1 = p_1$, $k'_i = p_i - p_{i-1} \, (2 \leq i \leq l)$. Extending by $\mathcal{C}$-linearity ϕ is defined as a map on $\mathfrak{h}^1$. Note that $\phi^2 = \mathrm{id}$.

Now we define a $\mathcal{C}$-linear map $\nabla_q : \mathcal{R}^{\mathbb{N}} \to \mathcal{R}^{\mathbb{N}}$ by

$$\nabla_q(b)(n) := \sum_{i=0}^{n} q^i \frac{(q^{-n})_i}{(q)_i} b(i),$$

where $(x)_n$ is the q-shifted factorial defined by (1.3). The following duality formula is due to Bradley [2] (see also [5]):

Proposition 2.1. *For $w \in \mathfrak{h}^1_{>0}$, we have*

$$\nabla_q(S_w)(n) = \begin{cases} 0 & (n = 0), \\ -\, s_{\phi(w)}(n-1) & (n \geq 1). \end{cases}$$

Remark 2.2. The operator ∇_q has another description. Consider the difference operator $\Delta_t : \mathcal{R}^{\mathbb{N}} \to \mathcal{R}^{\mathbb{N}}$ defined by $\Delta_t(b)(n) := b(n) - tb(n+1)$. Then we have $\nabla_q(b)(n) = (\Delta_{q^{-(n-1)}} \circ \cdots \circ \Delta_{q^{-1}} \circ \Delta_1(b))(0)$ (see Corollary 2.7 in [5]).

2.2. **A q-analogue of multiple zeta (star) values.** Introduce the valuation $v : \mathcal{R} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ defined by $v(f) := \inf\{j \,|\, c_j \neq 0\}$ for $f = \sum_{j=0}^{\infty} c_j q^j$, and endow $\mathcal{R}$ with the topology such that the sets $\{f + \phi \,|\, v(\phi) \geq n\}$ $(n = 0, 1, \ldots)$ form neighborhood base at $f \in \mathcal{R}$. Then $\mathcal{R}$ is complete.

Denote by $\mathfrak{h}^0$ the $\mathcal{C}$-submodule of $\mathfrak{h}^1$ generated by 1 and the words $z_{k_1} \cdots z_{k_r}$ satisfying $r \geq 1$ and $k_1 \geq 2$. If $w \in \mathfrak{h}^0$, the limit $\lim_{n\to\infty} A_w(n) = \sum_{k=0}^{\infty} a_w(k)$ converges in $\mathcal{R}$ since $v(a_w(n)) \geq n+1$. We call it a *q-analogue of multiple zeta value* (qMZV) and denote it by $\zeta_q(w)$. If w is a word $w = z_{k_1} \cdots z_{k_r} \in \mathfrak{h}^0$, it is given by

$$\zeta_q(z_{k_1} \cdots z_{k_r}) := \sum_{m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1 + (k_2-1)m_2 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}.$$

We also define a *q-analogue of multiple zeta star value* (qMZSV) (or that of *non-strict multiple zeta value*) by

$$\zeta_q^\star(z_{k_1} \cdots z_{k_r}) := \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{q^{(k_1-1)m_1 + (k_2-1)m_2 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}.$$

If we regard q as a complex variable, qMZV and qMZSV are absolutely convergent in $|q| < 1$. Therefore in fact they are convergent series. In the limit $q \to 1$, they converge to multiple zeta values (MZV) and multiple zeta star values (or non-strict multiple zeta values) defined by

$$\zeta(z_{k_1} \cdots z_{k_r}) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

and

$$\zeta^\star(z_{k_1} \cdots z_{k_r}) := \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}},$$

respectively.

2.3. **Algebraic structure.** Let $\mathfrak{z}$ be the $\mathcal{C}$-submodule of $\mathfrak{h}^1$ generated by $\{z_i\}_{i=1}^{\infty}$. We define three products $\circ, \circ_{\pm}$ on $\mathfrak{z}$ by setting

$$z_i \circ z_j := z_{i+j}, \qquad z_i \circ_{\pm} z_j := \pm z_{i+j} + \hbar z_{i+j-1}$$

and extending by $\mathcal{C}$-linearity. These products are associative and commutative.

Define two $\mathcal{C}$-bilinear products $*_{\pm}$ on $\mathfrak{h}^1$ inductively by

$$1 *_{\pm} w = w, \quad w *_{\pm} 1 = w,$$

$$(z_i w_1) *_{\pm} (z_j w_2) = z_i(w_1 *_{\pm} z_j w_2) + z_j(z_i w_1 *_{\pm} w_2) + (z_i \circ_{\pm} z_j)(w_1 *_{\pm} w_2) \tag{2.7}$$

for $i, j \geq 1$ and $w, w_1, w_2 \in \mathfrak{h}^1$. These products are commutative.

Proposition 2.3. *Let $w_1, w_2 \in \mathfrak{h}^1$. Then $S_{w_1} S_{w_2} = S_{w_1 *_- w_2}$ and $A_{w_1} A_{w_2} = A_{w_1 *_+ w_2}$.*

Proof. It suffices to show that $S_{w_1} S_{w_2}$ and $A_{w_1} A_{w_2}$ follow the recurrence relation (2.7) for $*_-$ and $*_+$, respectively. It can be checked by using

$$\frac{q^{2m}}{[m]^{i+j}} = \frac{q^m}{[m]^{i+j}} - (1-q)\frac{q^m}{[m]^{i+j-1}}$$

and

$$\frac{q^{(i+j-2)m}}{[m]^{i+j}} = \frac{q^{(i+j-1)m}}{[m]^{i+j}} + (1-q)\frac{q^{(i+j-2)m}}{[m]^{i+j-1}} \tag{2.8}$$

for $i, j \geq 1$. □

The products $\circ$ and $\circ_+$ determine $\mathfrak{z}$-module structures on $\mathfrak{h}^1$, which we denote by the same letters $\circ$ and $\circ_+$, such that

$$z_i \circ 1 := 0, \qquad z_i \circ (z_j w) := (z_i \circ z_j) w \quad (i, j \geq 1,\ w \in \mathfrak{h}^1)$$

and the above formulas where $\circ$ is replaced with $\circ_{\pm}$. By convention we set $z_0 \circ c = 0$ for $c \in \mathcal{C}$ and $z_0 \circ w = w$ for $w \in \mathfrak{h}^1_{>0}$. Then $z_i \circ_+ w = (z_i + \hbar z_{i-1}) \circ w$ for $i \geq 1$ and $w \in \mathfrak{h}^1$.

Consider the $\mathcal{C}$-linear map d_q on $\mathfrak{h}^1$ defined inductively by

$$d_q(1) = 1, \qquad d_q(z_i w) = z_i\, d_q(w) + z_i \circ_+ d_q(w) \quad (i \geq 1,\ w \in \mathfrak{h}^1).$$

Note that d_q is invertible and its inverse satisfies $d_q^{-1}(1) = 1$ and

$$d_q^{-1}(z_i w) = z_i\, d_q^{-1}(w) - z_i \circ_+ d_q^{-1}(w) \quad (i \geq 1,\ w \in \mathfrak{h}^1).$$

Lemma 2.4. *For $i \geq 1$ and $w \in \mathfrak{h}^1$, we have $d_q(z_i \circ w) = z_i \circ d_q(w)$, that is, the map d_q commutes with the action $\circ$ of $\mathfrak{z}$.*

Proof. For $w = 1$ it is trivial. Suppose that w is a word and set $w = z_j w'$. Then we have $d_q(z_i \circ w) = d_q(z_{i+j} w') = z_{i+j}\, d_q(w') + z_{i+j} \circ_+ d_q(w') = z_i \circ (z_j d_q(w')) + (z_i \circ z_j) \circ_+ d_q(w')$. Using $(z_i \circ z_j) \circ_+ z_k = z_i \circ (z_j \circ_+ z_k)$ for $i, j, k \geq 1$, we see that $(z_i \circ z_j) \circ_+ d_q(w') = z_i \circ (z_j \circ_+ d_q(w'))$. Thus we get $d_q(z_i \circ w) = z_i \circ d_q(w)$ for a word w. From the $\mathcal{C}$-linearity of d_q, we obtain the lemma. □

Proposition 2.5. *Let $w \in \mathfrak{h}^1$. Then $A^{\star}_w = A_{d_q(w)}$.*

Proof. It is enough to prove the case where w is a word. We prove the proposition by induction on the depth of w. If $\mathrm{dep}(w)=1$, that is, $w=z_i$ for some $i\geq 1$, we have $A^{\star}_{z_i}=A_{z_i}=A_{d_q(z_i)}$.

From the definition of A and $A^{\star}$ we find that

$$\sum_{m=1}^{n}\frac{q^{(i-1)m}}{[m]^i}A^{\star}_{w}(m)=A^{\star}_{z_iw}(n),$$
$$\sum_{m=1}^{n}\frac{q^{(i-1)m}}{[m]^i}A_{w}(m)=A_{z_iw+z_i\circ_+ w}(n)$$

for $w\in\mathfrak{h}^1$. To show the second formula, divide the sum in the definition (2.2) of $A_w(m)$ into two parts with $m_1=m$ and with $m_1<m$, and use (2.8) for the first part. Now suppose that $\mathrm{dep}(w)>1$ and the proposition holds for words whose depth is less than $\mathrm{dep}(w)$. Then setting $w=z_iw'$, we find

$$\begin{aligned}A^{\star}_{w}(n)&=\sum_{m=1}^{n}\frac{q^{(i-1)m}}{[m]^i}A^{\star}_{w'}(m)=\sum_{m=1}^{n}\frac{q^{(i-1)m}}{[m]^i}A_{d_q(w')}(m)\\&=A_{z_id_q(w')+z_i\circ_+ d_q(w')}(n)=A_{d_q(w)}(n).\end{aligned}$$

□

Corollary 2.6. *For $w\in\mathfrak{h}^1$, we have $\zeta_q^{\star}(w)=\zeta_q(d_q(w))$.*

We define a $\mathcal{C}$-bilinear product $\circledast_q$ on $\mathfrak{h}^1_{>0}$ by

$$(z_iw_1)\circledast_q(z_jw_2):=(z_i\circ z_j)(w_1*_+ w_2)\qquad(i,j\geq 1,\ w_1,w_2\in\mathfrak{h}^1). \tag{2.9}$$

It is commutative and associative.

Proposition 2.7. *Let $w_1,w_2\in\mathfrak{h}^1_{>0}$. Then $s_{w_1}a_{w_2}=a_{d_q(w_1)\circledast_q w_2}$.*

To prove Proposition 2.7 we need the following lemma:

Lemma 2.8. *For $n\in\mathbb{N}$ and $w\in\mathfrak{h}^1$, we have*

$$\frac{q^{(i-1)(n+1)}}{[n+1]^i}A_w(n+1)=a_{z_iw+z_i\circ_+ w}(n). \tag{2.10}$$

Proof. It is trivial for $w=1$. Without loss of generality we can assume that w is a word. Set $w=z_jw'$. Then we have

$$A_w(n+1)=\frac{q^{(j-1)(n+1)}}{[n+1]^j}A_{w'}(n)+A_w(n).$$

Substitute it into the left hand side of (2.10) and use (2.8). Then we obtain

$$\left(\frac{q^{(i+j-1)(n+1)}}{[n+1]^{i+j}}+(1-q)\frac{q^{(i+j-2)(n+1)}}{[n+1]^{i+j-1}}\right)A_{w'}(n)+\frac{q^{(i-1)(n+1)}}{[n+1]^i}A_{w'}(n).$$

Using (2.6) again, we see that it is equal to the right hand side of (2.10). □

Proof of Proposition 2.7. We can assume that w_1 and w_2 are words. Set $w_1 = z_i w_1'$ and $w_2 = z_j w_2'$. Using (2.6) and Proposition 2.5, we have

$$(s_{w_1} a_{w_2})(n) = \frac{q^{(i+j-1)(n+1)}}{[n+1]^{i+j}} A_{d_q(w_1')}(n+1) A_{w_2'}(n).$$

It is equal to $a_{d_q(z_{i+j}w_1')}(n) A_{w_2'}(n)$ because of Lemma 2.8 and the definition of d_q. Now introduce a $\mathcal{C}$-bilinear map $\triangle : \mathfrak{h}^1_{>0} \times \mathfrak{h}^1 \to \mathfrak{h}^1_{>0}$ uniquely determined from the property

$$(z_i w')\triangle w'' := z_i(w' *_+ w'') \qquad (i \geq 1).$$

Then $a_w A_{w''} = a_{w \triangle w''}$ for $w \in \mathfrak{h}^1_{>0}$ and $w'' \in \mathfrak{h}^1$ because of (2.6) and Proposition 2.3. Therefore Proposition 2.7 is reduced to the identity

$$(d_q(z_{i+j}w_1'))\, \triangle w_2' = (d_q(z_i w_1'))\circledast_q (z_j w_2') \tag{2.11}$$

for $i, j \geq 1$ and $w_1', w_2' \in \mathfrak{h}^1$.

Let us prove (2.11). From the definition of d_q and $\triangle$, the left hand side is equal to

$$z_{i+j}(d_q(w_1') *_+ w_2') + (z_{i+j} \circ_+ d_q(w_1'))\triangle w_2'.$$

The first term is equal to $(z_i d_q(w_1'))\circledast_q (z_j w_2')$ because $z_{i+j} = z_i \circ z_j$. Now note that $z_{i+j} \circ_+ z_k = (z_i \circ_+ z_k) \circ z_j$ for any $i, j, k \geq 1$. Using this we see that the second term is equal to $(z_i \circ_+ d_q(w_1'))\circledast_q (z_j w_2')$. Summing up these two terms we get the right hand side of (2.11). □

3. A q-analogue of Newton series

We consider the ring of formal power series $\mathcal{R}[[z]] = \mathbb{Q}[[q, z]]$. Introduce the valuation on $\mathcal{R}[[z]]$ by $v(f) = \inf\{i + j \mid c_{ij} \neq 0\}$ for $f = \sum_{i,j=0}^{\infty} c_{ij} q^i z^j$ and endow $\mathcal{R}[[z]]$ with the structure of topological $\mathbb{Q}$-algebra.

Define polynomials $B_n(z)\,(n \in \mathbb{N})$ by

$$B_0(z) := 1, \qquad B_n(z) := z^n \frac{(z^{-1})_n}{(q)_n} = \prod_{j=1}^{n} \frac{z - q^{j-1}}{1 - q^j} \quad (n \geq 1).$$

Proposition 3.1. *Let $b \in \mathcal{R}^{\mathbb{N}}$ and $m, l \in \mathbb{N}$. Then*

$$\sum_{n=0}^{m} \nabla(b)(l+n)\, B_n(q^m) = q^{-lm} \sum_{j=0}^{l} q^j \frac{(q^{-l})_j}{(q)_j}\, b(j+m). \tag{3.1}$$

In particular, we have $\sum_{n=0}^{m} \nabla(b)(n) B_n(q^m) = b(m)$.

In the proof of Proposition 3.1 we use the following formula:

Lemma 3.2. *For $m, j \in \mathbb{N}$ we have*

$$\frac{1}{(q)_j}\sum_{n=0}^{m} q^{mn}\frac{(q^{-m})_n(tq^{-n})_j}{(q)_n} = \begin{cases} 0 & (0 \le j < m), \\ (q^{-1}t)^m \dfrac{(t)_{j-m}}{(q)_{j-m}} & (m \le j). \end{cases}$$

Proof. Multiply the both sides by s^j and sum up over $j \in \mathbb{N}$ using the q-binomial formula

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n = \frac{(ax)_\infty}{(x)_\infty}. \tag{3.2}$$

Then we see that the identity to prove is equivalent to

$$\sum_{n=0}^{m} q^{mn}\frac{(q^{-m})_n}{(q)_n}(q^{-n}st)_n = (q^{-1}st)^m. \tag{3.3}$$

Using

$$(q^{-a}x)_n = (-x)^n q^{-an+n(n-1)/2}(q^{a-n+1}x^{-1})_n, \tag{3.4}$$

we find that the left hand side of (3.3) is equal to

$$(q)_m\sum_{n=0}^{m}\frac{(qs^{-1}t^{-1})_n}{(q)_n(q)_{m-n}}(q^{-1}st)^n.$$

It is equal to the coefficient of x^m in

$$(q)_m\left(\sum_{i=0}^{\infty}(q^{-1}stx)^i\frac{(qs^{-1}t^{-1})_i}{(q)_i}\right)\left(\sum_{i=0}^{\infty}\frac{x^i}{(q)_i}\right),$$

and the above product is equal to

$$(q)_m\frac{(x)_\infty}{(q^{-1}stx)_\infty}\frac{1}{(x)_\infty} = (q)_m\frac{1}{(q^{-1}stx)_\infty} = (q)_m\sum_{j=0}^{\infty}\frac{(q^{-1}stx)^j}{(q)_j}$$

from the q-binomial formula (3.2). Thus we obtain the right hand side of (3.3). □

Proof of Proposition 3.1. From $(q^{-l-n})_j = 0$ for $j > l+n$, we have

$$\sum_{n=0}^{m}\nabla(b)(l+n)\,B_n(q^m) = \sum_{j=0}^{l+m}\frac{q^j}{(q)_j}\,b(j)\sum_{n=0}^{m}q^{mn}\frac{(q^{-m})_n(q^{-l-n})_j}{(q)_n}.$$

The second sum in the right hand side can be calculated by using Lemma 3.2 with $t = q^{-l}$. Then we obtain the right hand side of (3.1). □

Suppose that $c \in \mathcal{R}^{\mathbb{N}}$ satisfies

$$v(c(n)) \ge n \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Set

$$f_c(z) := \sum_{n=0}^{\infty} c(n)\, B_n(z). \tag{3.6}$$

Then the series $f_c(z)$ converges in $\mathcal{R}[[z]]$. If $c = \nabla_q(b)$ satisfies the condition (3.5), the series $f_{\nabla_q(b)}$ is well-defined and we have $f_{\nabla_q(b)}(q^m) = b(m)$ for all $m \in \mathbb{N}$ from Lemma 3.1. Thus we may regard the series $f_{\nabla_q(b)}(z)$ as a q-analogue of Newton series interpolating $b \in \mathcal{R}^{\mathbb{N}}$.

Let us prove two properties of the series f_c.

Proposition 3.3. *Suppose that $c \in \mathcal{R}^{\mathbb{N}}$ satisfies the condition* (3.5). *Then the following equality holds in $\mathcal{R}[[z]]$:*

$$f_c(z) = c(0) + \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} c(n)\, a_{z_1^m}(n-1) \right) \left(\frac{z-1}{1-q} \right)^m,$$

where $a_{z_1^m}(n)$ is defined by (2.5).

Proof. It follows from

$$B_n(z) = \frac{1}{[n]} \frac{z-1}{1-q} \prod_{j=1}^{n-1} \left(1 + \frac{1}{[j]} \frac{z-1}{1-q} \right) = \sum_{m=1}^{n} a_{z_1^m}(n-1) \left(\frac{z-1}{1-q} \right)^m$$

and $a_{z_1^m}(k-1) = 0$ for $k < m$. □

Proposition 3.4. *Suppose that $\nabla_q(b_i) \in \mathcal{R}^{\mathbb{N}}\, (i = 1, 2)$ satisfy the condition* (3.5). *Then we have $f_{\nabla_q(b_1)}(z) f_{\nabla_q(b_2)}(z) = f_{\nabla_q(b_1 b_2)}(z)$.*

We prove two lemmas to show Proposition 3.4.

Lemma 3.5. *For $n \in \mathbb{N}$ we have*

$$B_n(z) = y^n \sum_{j=0}^{n} \frac{(y^{-1})_{n-j}}{(q)_{n-j}} B_j(y^{-1} z). \tag{3.7}$$

Proof. The right hand side is equal to the coefficient of t^n in

$$\left(\sum_{j=0}^{\infty} (ty)^j \frac{(y^{-1})_j}{(q)_j} \right) \left(\sum_{j=0}^{\infty} (tz)^j \frac{(yz^{-1})_j}{(q)_j} \right).$$

From the q-binomial formula (3.2), the above product is equal to

$$\frac{(t)_\infty}{(yt)_\infty} \frac{(yt)_\infty}{(tz)_\infty} = \frac{(t)_\infty}{(tz)_\infty} = \sum_{n=0}^{\infty} t^n B_n(z).$$

This completes the proof. □

Lemma 3.6. *Suppose that $c_1, c_2 \in \mathcal{R}^{\mathbb{N}}$ satisfy the condition* (3.5)*. Set $c_3 \in \mathcal{R}^{\mathbb{N}}$ by*

$$c_3(n) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q c_1(k) \sum_{j=0}^{k} c_2(n-k+j) B_j(q^k),$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

Then we have $f_{c_1}(z) f_{c_2}(z) = f_{c_3}(z)$.

Proof. Substitute (3.7) into $f_{c_1}(z) f_{c_2}(z) = \sum_{m,n=0}^{\infty} c_1(m) c_2(n) B_m(z) B_n(z)$ setting $y = q^m$. Using $B_m(z) B_j(q^{-m} z) = q^{-jm} B_{m+j}(z)$, we find that

$$f_{c_1}(z) f_{c_2}(z) = \sum_{m,n=0}^{\infty} \sum_{j=0}^{n} c_1(m) c_2(n) q^{(n-j)m} \frac{(q^{-m})_{n-j}}{(q)_{n-j}} \begin{bmatrix} m+j \\ m \end{bmatrix}_q B_{m+j}(z).$$

Then we can see that it is equal to $f_{c_3}(z)$. □

Proof of Proposition 3.4. From Proposition 3.1 and Lemma 3.6, it suffices to prove that

$$\sum_{0 \le i \le j \le n} \frac{q^{i+j}}{(q)_i} b_1(i)\, b_2(j) \sum_{k=i}^{j} q^{k(k-n-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{-k})_i (q^{-n+k})_{j-k}}{(q)_{j-k}} = \nabla(b_1 b_2)(n). \tag{3.8}$$

Using (3.4), we see that the second sum in the left hand side above is equal to

$$(-1)^j q^{j(j-1)/2 - nj - i} \frac{(q)_n}{(q)_{n-j} (q)_{j-i}} \sum_{k=0}^{j-i} (-1)^k q^{k(k-1)/2} \begin{bmatrix} j-i \\ k \end{bmatrix}_q.$$

For $N \in \mathbb{N}$, we have

$$\sum_{k=0}^{N} (-1)^k q^{k(k-1)/2} \begin{bmatrix} N \\ k \end{bmatrix}_q = \delta_{N,0}.$$

Thus we find that the left hand side of (3.8) is equal to

$$\sum_{i=0}^{n} (-1)^i \frac{q^{-ni+i(i+1)/2}}{(q)_i} \frac{(q)_n}{(q)_{n-i}} b_1(i)\, b_2(i) = \sum_{i=0}^{n} q^i \frac{(q^{-n})_i}{(q)_i} (b_1 b_2)(i) = \nabla(b_1 b_2)(n).$$

□

4. A q-analogue of Kawashima's relation

4.1. **Kawashima's relation.** Define a product $\overline{*}$ on $\mathfrak{h}^1$ inductively by

$$1 \,\overline{*}\, w = w, \quad w \,\overline{*}\, 1 = w,$$
$$(z_i w_1) \,\overline{*}\, (z_j w_2) = z_i(w_1 \,\overline{*}\, z_j w_2) + z_j(z_i w_1 \,\overline{*}\, w_2) - z_{i+j}(w_1 \,\overline{*}\, w_2)$$

for $i, j \geq 1$ and $w, w_1, w_2 \in \mathfrak{h}^1$. We also define a product $\circledast$ on $\mathfrak{h}^1_{>0}$ by the formula (2.9) where the product $*_+$ in the right hand side is replaced with $*$. We define a $\mathcal{C}$-linear map $d : \mathfrak{h}^1 \to \mathfrak{h}^1$ by the properties $d(1) = 1$ and $d(z_i w) = z_i\, d(w) + z_i \circ d(w)$ for $w \in \mathfrak{h}^1$. The products $\overline{*}, \circledast$ and the map d can be regarded as the limit of $*_-, \circledast_q$ and d_q as $\hbar \to 0$ (or $q \to 1$), respectively.

Kawashima proved the following relation for MZV's [4]:

(4.1)
$$\zeta(d(\phi(w_1 \,\overline{*}\, w_2)) \circledast z_1^n) + \sum_{\substack{k+l=n \\ k,l \geq 1}} \zeta(d(\phi(w_1)) \circledast z_1^k)\, \zeta(d(\phi(w_2)) \circledast z_1^l) = 0 \quad (w_1, w_2 \in \mathfrak{h}^1_{>0}).$$

Setting $n = 1$ we obtain linear relations for MZV's

$$\zeta(z_1 \circ d(\phi(w_1 \,\overline{*}\, w_2))) = 0 \qquad (w_1, w_2 \in \mathfrak{h}^1_{>0}). \tag{4.2}$$

4.2. **A q-analogue of Kawashima's relation.**

Proposition 4.1. *For $w_1, w_2 \in \mathfrak{h}^1_{>0}$ and $n \geq 1$, the following relation holds:*

(4.3)
$$\zeta_q(d_q(\phi(w_1 *_- w_2)) \circledast_q z_1^n) + \sum_{\substack{k+l=n \\ k,l \geq 1}} \zeta_q(d_q(\phi(w_1)) \circledast_q z_1^k)\, \zeta_q(d_q(\phi(w_2)) \circledast_q z_1^l) = 0.$$

Proof. If $w \in \mathfrak{h}^1_{>0}$, we have $\nabla_q(S_w)(0) = 0$ and

$$v(\nabla_q(S_w)(n)) = v(-s_{\phi(w)}(n-1)) \geq n \qquad (n > 0).$$

from Proposition 2.1. Hence $\nabla_q(S_w)$ satisfies the condition (3.5), and the series $F_w(z) := f_{\nabla(S_w)}(z)$ is well-defined. Expanding $F_w(z)$ at $z = 1$ using Proposition 3.3 and Proposition 2.7, we see that

$$\begin{aligned} F_w(z) &= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \nabla(S_w)(n)\, a_{z_1^m}(n-1) \right) \left(\frac{z-1}{1-q} \right)^m \\ &= -\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} s_{\phi(w)}(n-1)\, a_{z_1^m}(n-1) \right) \left(\frac{z-1}{1-q} \right)^m \\ &= -\sum_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} a_{d_q(\phi(w)) \circledast_q z_1^m}(n) \right) \left(\frac{z-1}{1-q} \right)^m \\ &= -\sum_{m=1}^{\infty} \zeta_q(d_q(\phi(w)) \circledast_q z_1^m) \left(\frac{z-1}{1-q} \right)^m. \end{aligned} \tag{4.4}$$

From Lemma 2.3 and Proposition 3.4, we have $F_{w_1}F_{w_2}=F_{w_1 *_- w_2}$ for $w_1, w_2 \in \mathfrak{h}^1_{>0}$. Substituting (4.4), we get the relation (4.3). □

Corollary 4.2. *For $w_1, w_2 \in \mathfrak{h}^1_{>0}$, we have*

$$\zeta_q^\star(z_1 \circ \phi(w_1 *_- w_2)) = 0.$$

Proof. Set $n=1$ in (4.3). Then it follows from Lemma 2.4 and Corollary 2.6. □

Let us rewrite the quadratic relation (4.3) into a similar form to Kawashima's relation (4.1). Consider the map

$$\Psi := \phi\, d_q^{-1} d\, \phi\, : \mathfrak{h}^1 \to \mathfrak{h}^1.$$

Note that $\Psi(1)=1$.

Lemma 4.3. *Let $w \in \mathfrak{h}^1$. Then*

$$\begin{aligned}
&\Psi(z_1 w) = z_1 \Psi(w), \\
&\Psi(z_i w) = (z_1 - \hbar z_0) \circ \Psi(z_{i-1} w) \qquad (i \geq 2).
\end{aligned}$$

Proof. Note that $\phi(z_i w) = z_1^{i-1}(z_1 \circ \phi(w))$ for $i \geq 1$ and $w \in \mathfrak{h}^1$. Since the maps d_q and d commute with the action $\circ$ of $\mathfrak{z}$, we get $\Psi(z_1 w) = z_1 \Psi(w)$. To show the second formula, use the identity

$$(d_q^{-1} d)(z_1 w) = (z_1 - \hbar z_0) \circ (d_q^{-1} d)(w) \qquad (w \in \mathfrak{h}^1)$$

following from the definition of d_q and d. □

Proposition 4.4. *Let $w \in \mathfrak{h}^1$. Then $\Psi(z_i w) = \xi_i \Psi(w)$ for $i \geq 1$, where $\xi_i \in \mathfrak{z}$ is given by*

$$\xi_i := \sum_{k=0}^{i-1} \binom{i-1}{k} (-\hbar)^{i-1-k} z_{k+1}.$$

Proof. Proposition 4.3 implies that it suffices to prove $(z_1 - \hbar z_0) \circ \xi_i = \xi_{i+1}$ for $i \geq 1$. This can be checked by direct calculation. □

Proposition 4.5. *Let $w_1, w_2 \in \mathfrak{h}^1$. Then*

$$\Psi(w_1) *_- \Psi(w_2) = \Psi(w_1 \mathbin{\overline{*}} w_2).$$

Proof. We can assume that w_1 and w_2 are words. Let us prove the proposition by induction on the depth of w_1 and w_2. If $w_1 = 1$ or $w_2 = 1$, it is trivial. Set $w_1 = z_i w_1'$ and $w_2 = z_j w_2'$. From Proposition 4.4 and the induction hypothesis, we find

$$\begin{aligned}
\Psi(w_1) *_- \Psi(w_2) &= (\xi_i \Psi(w_1')) *_- (\xi_j \Psi(w_2')) \\
&= \xi_i \Psi(w_1' \mathbin{\overline{*}} w_2) + \xi_j \Psi(w_1 \mathbin{\overline{*}} w_2') + (\xi_i \circ_- \xi_j) \Psi(w_1' \mathbin{\overline{*}} w_2').
\end{aligned}$$

By direct calculation we see that $\xi_i \circ_- \xi_j = -\xi_{i+j}$ for $i, j \geq 1$. Therefore we get

$$\begin{aligned}\Psi(w_1) *_- \Psi(w_2) &= \xi_i \Psi(w_1' \,\overline{*}\, w_2) + \xi_j \Psi(w_1 \,\overline{*}\, w_2') - \xi_{i+j} \Psi(w_1' \,\overline{*}\, w_2') \\ &= \Psi(z_i(w_1' \,\overline{*}\, w_2) + z_j(w_1 \,\overline{*}\, w_2') - z_{i+j}(w_1' \,\overline{*}\, w_2')) \\ &= \Psi(w_1 \,\overline{*}\, w_2).\end{aligned}$$

□

Now we are ready to derive a q-analogue of Kawashima's relations:

Theorem 4.6. *For $w_1, w_2 \in \mathfrak{h}^1_{>0}$ and $n \geq 1$, the following relation holds:*

$$\zeta_q(d(\phi(w_1 \,\overline{*}\, w_2)) \circledast_q z_1^n) + \sum_{\substack{k+l=n \\ k,l \geq 1}} \zeta_q(d(\phi(w_1)) \circledast_q z_1^k)\, \zeta_q(d(\phi(w_2)) \circledast_q z_1^l) = 0. \tag{4.5}$$

Proof. Substitute $\Psi(w_i)$ into w_i $(i = 1, 2)$ in the quadratic relation (4.3). Then we get the relation (4.5) using Proposition 4.5 and $d_q \phi \Psi = d\phi$. □

Setting $n = 1$ in (4.5), we obtain linear relations for qMZV's in the same form as (4.2):

Corollary 4.7. *For $w_1, w_2 \in \mathfrak{h}^1_{>0}$, we have*

$$\zeta_q(z_1 \circ d(\phi(w_1 \,\overline{*}\, w_2))) = 0.$$

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